JOURNAL OF COMPUTATIONAL PHYSICS 107, 254–261 (1993)

Pole Condition for Singular Problems: The Pseudospectral Approximation

WEIZHANG HUANG*

Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, Scotland and Institute of Applied Mathematics, Academia Sinica, Beijing 100080, China

AND

DAVID M. SLOAN

Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, Scotland

Received February 14, 1992; revised May 22, 1992

This paper deals with the pseudospectral solution of differential equations with coordinate singularities such as those which describe situations in spherical or cylindrical geometries. We use the differential equation, together with a smoothness assumption on the solution, to construct "pole conditions." The pole conditions, which are straightforward and easily implemented, serve as numerical boundary conditions at the coordinate singularity. Standard pseudospectral methods, including fast transformation techniques, can then be applied to the singular problem. The method is illustrated using the eigenvalue problem of Bessel's equation and a Poisson equation on the unit disk. Numerical results show that spectral convergence is achieved.

1. INTRODUCTION

Many physical situations give rise to mathematical models involving singular differential problems with smooth solutions. For example, solutions of differential equations in cylindrical or spherical geometries have special behaviour near the coordinate singularities and this forces the solutions to be smooth. Numerical methods for approximating solutions of problems with coordinate singularities should be designed to capture the special behaviour of the exact differential solutions at the singularities. Here we are concerned with the solution of problems with coordinate singularities by means of pseudospectral methods.

When spectral methods are applied to many singular problems convergent solutions can be obtained, even when the singularity has not been treated with special care. However, the accuracy or computational efficiency may be degraded since standard spectral representations either do not fully capture the behaviour of solutions near the singularities, or they are ill-suited to fast transform techniques [1–3, 5]. Several approaches, such as expansions in spherical harmonics, parity-modified Fourier series and modified Robert functions [2], or specially chosen basis functions [4], have been presented in attempts to capture the special behaviour of solutions near coordinate singularities. They nearly all follow the same pattern in seeking an approximation expanded in certain special functions which satisfy some "pole conditions" derived from the smoothness of solutions. These expansions work well in spectral Galerkin and spectral tau methods. They are not well-suited to pseudospectral (interpolatory) methods, which is unfortunate, since these methods are more readily applied to variable coefficient and nonlinear problems.

In this paper we deal with the application of pseudospectral methods to singular problems using an approach which is related to that adopted by Gottlieb and Orszag [5, pp. 152–153] in their Chebyshev tau solution of Bessel's equation. We shall use the differential equation to construct additional pole conditions which will be simple and easily used. The extra pole conditions will serve as numerical boundary conditions which will enable us to solve singular problems by *standard* pseudospectral methods.

In Section 2 of the paper we consider as a one-dimensional example the computation of eigenvalues of Bessel's equation. The example is used to illustrate how additional pole conditions are constructed using the differential equation. Numerical results for the eigenproblem are also presented in this section. Section 3 shows how pole conditions are obtained for a Poisson-type equation on a circular disk. The solution of this problem by pseudospectral

* Supported by a Royal Fellowship Award from the Royal Society of London.

0021-9991/93 $5.00

methods is discussed in Section 4, and a finite-difference preconditioner for the pseudospectral differentiation matrix is also considered. Section 5 contains conclusions and comments.

Throughout this paper, any problem considered is assumed to have a unique and sufficiently smooth solution.

2. EIGENVALUE PROBLEM OF BESSEL'S EQUATION

As a simple example of singular problems, let us consider the spectral computation of the eigenvalues of Bessel's equation, which has been discussed by Gottlieb and Orszag [5, pp. 152–153]. The problem is to find the eigenvalues, λ, and eigenfunctions, $u(x)$, of

$$u'' + \frac{1}{x} u' - \frac{n^2}{x^2} u = -\lambda u, \tag{2.1}$$

subject to the conditions that

$$u(1) = 0 \tag{2.2}$$

and that $u(x)$ be finite for $0 \leqslant x \leqslant 1$. The exact eigenvalues are related to the zeros of the Bessel function J_n and are given by $\lambda_{np} = j_{np}^2$, where $J_n(j_{np}) = 0$, $p = 1, 2, \ldots$.

When n is even the eigenfunctions of (2.1) are even functions of x, and when n is odd the eigenfunctions are odd. For odd n, this parity property enables us to approximate $u(x)$ on $[-1, 1]$ by

$$u^{2M-1}(x) = \sum_{m=1}^{M} \hat{u}_m T_{2m-1}(x), \tag{2.3}$$

where $T_{2m-1}(x)$ is a Chebyshev polynomial of degree $2m-1$. Table 14.1 in [5] lists numerical results for the smallest eigenvalue, λ_{71}, of (2.1) with $n = 7$, obtained using (2.2), (2.3), and the Chebyshev tau method. The convergence of this method, albeit very impressive as M increases, is degraded by the coordinate singularity of (2.1) at $x = 0$. Gottlieb and Orszag [5] also showed that it is possible to improve the convergence of (2.3) by imposing additional "pole conditions," like

$$u'(0) = 0. \tag{2.4}$$

The numerical values of the smallest eigenvalue, λ_{71}, of (2.1) with $n = 7$, obtained using (2.3), (2.2), (2.4), and the Chebyshev tau method are also listed in Table 14.1 in [5]. There is clearly a dramatic improvement in the rate of convergence with (2.4) applied.

However, there are several difficulties in using (2.3) and (2.4) in a pure "interpolatory" method—the pseudospectral method. First, the even–odd property of $u(x)$ associated with the parity of n is not well-suited to the pseudospectral method. Moreover, it is not easy to find and to treat the parity of solutions of higher-dimensional problems. Finally, there is a question of whether (2.4) is correct for (2.1) with all non-negative integer values of n.

To show the suitability, or otherwise, of (2.4) for $n = 0, 1, 2, \ldots$, let us consider the pseudospectral approximation to (2.1) in detail. Define the transformed Chebyshev–Lobatto points as

$$x_k = \frac{1 - \cos(k\pi/N)}{2}, \qquad k = 0, 1, \ldots, N, \tag{2.5}$$

where N is a certain positive integer. Let

$$u^N(x) = \sum_{k=0}^{N} u_k h_k(x), \tag{2.6}$$

where

$$h_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{N} \frac{x - x_i}{x_k - x_i}. \tag{2.7}$$

The collocation equations are defined by

$$\frac{d^2 u^N}{dx^2}(x_j) + \frac{1}{x_j} \frac{du^N}{dx}(x_j) - \frac{n^2}{x_j^2} u^N(x_j) = -\lambda u^N(x_j), \qquad 1 \leqslant j \leqslant N-1, \tag{2.8}$$

subject to the given boundary condition (2.2) at $x = 1$,

$$u^N(1) = 0, \tag{2.9}$$

and the pole condition (as a boundary condition)

$$\frac{du^N}{dx}(0) = 0. \tag{2.10}$$

As functions of N, relative errors, RE, of computed values of the two smallest eigenvalues obtained using (2.8)–(2.10) are plotted in Fig. 2.1 for the cases $n = 0, 1, 2, 7$. Figure 2.1 shows that the imposition of (2.4) as a boundary condition at the singular point $x = 0$ actually gives spectral accuracy for all cases except $n = 1$. The computation for the case $n = 1$

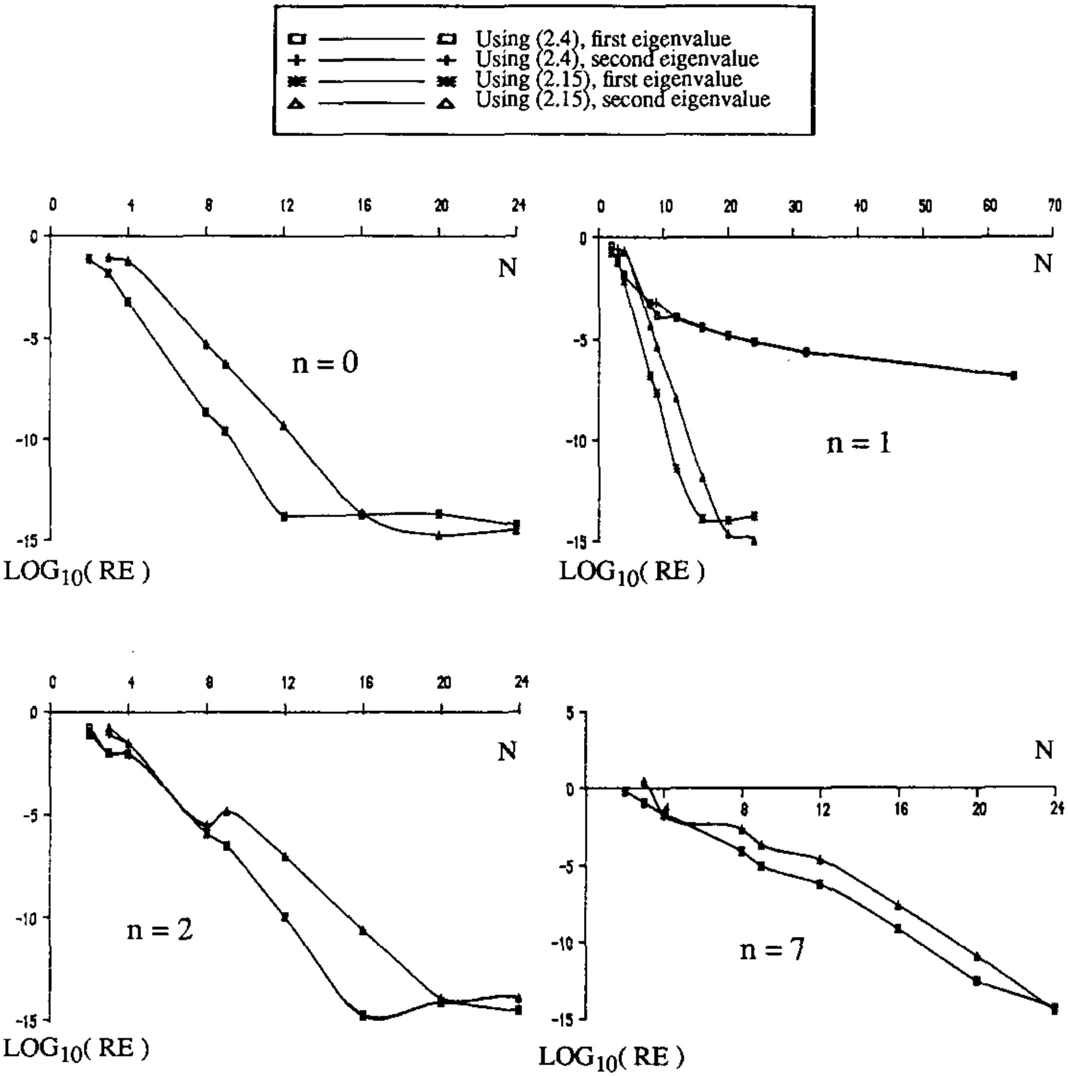

FIG. 2.1. Relative errors of computed smallest two eigenvalues of (2.1) as functions of N. Pole condition, either (2.4) or (2.15), is used in the pseudospectral approximation.

suffers a deleterious effect from the singularity. The numerical approximation in this case has not captured the behaviour of the solution near $x = 0$. This can be illustrated in the following analysis.

If we assume that (2.1) admits a solution $u \in C^2[0, 1]$, (2.1) implies that

$$\frac{d^2u}{dx^2}(x) + \frac{1}{x}\left[\frac{du}{dx}(0) + x\frac{d^2u}{dx^2}(\eta)\right] - \frac{n^2}{x^2}\left[u(0) + x\frac{du}{dx}(0) + \frac{x^2}{2}\frac{d^2u}{dx^2}(\xi)\right] = -\lambda u(x), \tag{2.11}$$

where $0 < \eta,\ \xi < x < 1$. Taking the limit as $x \to 0$ we obtain

$$O\left(\frac{1}{x^2}\right): \ -n^2 u(0) = 0, \tag{2.12}$$

$$O\left(\frac{1}{x}\right): (1 - n^2)\frac{du}{dx}(0) = 0, \tag{2.13}$$

$$O(1): \left(2 - \frac{n^2}{2}\right)\frac{d^2u}{dx^2}(0) = -\lambda u(0). \tag{2.14}$$

It is worth noting that (2.12)–(2.14) are derived from the differential equation (2.1) itself and the assumption of smoothness of the solution. Therefore, (2.12)–(2.14) should contain sufficient information to handle the special behaviour of the solution near the coordinate singularity. Indeed, (2.12)–(2.14) contain the pole condition (2.4) for cases $n \neq 1$ in which spectral accuracy can be obtained by imposing

(2.4) as a boundary condition. Moreover, they do not give (2.4) for the case $n=1$ in which the pseudospectral approximation using (2.4) has poor convergence. This illustrates why imposition of (2.4) cannot improve the rate of convergence for the case $n=1$.

It is obvious that use of all three conditions (2.12)–(2.14) creates an overdetermined system. Actually, it is not necessary to impose all these constraints to obtain very high accuracy. From the orders of $1/x$ in (2.12)–(2.14) it is natural to choose (2.12). But (2.12) suffers also from the "defect" at $n=0$. Therefore, we should add one "rank" to (2.12) from (2.13) when $n=0$. This gives

$$-n^2u(0)+\chi(-n^2)\frac{du}{dx}(0)=0, \tag{2.15}$$

where the function $\chi(a)$ is defined by

$$\chi(a)=\begin{cases}1, & a=0,\\ 0, & a\neq 0.\end{cases} \tag{2.16}$$

Obviously, (2.15) is simple to use in the pseudospectral approximation. The collocation equation of (2.15) is given by

$$-n^2u^N(0)+\chi(-n^2)\frac{du^N}{dx}(0)=0. \tag{2.17}$$

RE of computed values of the two smallest eigenvalues obtained using (2.8), (2.9), and (2.17) are also plotted as functions of N in Fig. 2.1. It is shown in Fig. 2.1 that the pseudospectral method imposing (2.15) gives spectral accuracy for all cases $n=0, 1, 2, 7$ and it has nearly the same accuracy as that given by imposing (2.4) for cases $n=0, 2, 7$.

3. POLE CONDITIONS FOR POISSON-TYPE EQUATIONS ON THE UNIT DISK

In this section we show how to construct pole conditions for Poisson-type equations on the unit disk following the approach discussed in Section 2. Consider

$$\begin{aligned}-\Delta u+\lambda u&=f, \qquad 0<r<1, \quad 0\leqslant\theta<2\pi,\\ u(1,\theta)&=g(\theta),\end{aligned} \tag{3.1}$$

where λ denotes a real constant, $g(\theta)$ is a given 2π-periodic function, and Δ denotes Laplace's operator in polar coordinates,

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{3.2}$$

Equations (3.1) and the assumption that $\partial^2u/\partial r^2$, $\partial^2u/\partial\theta^2$, and $\partial^4u/\partial r^2\,\partial\theta^2$ are continuous in the unit disk imply that

$$\begin{aligned}&-\frac{\partial^2u}{\partial r^2}(r,\theta)-\frac{1}{r}\left[\frac{\partial u}{\partial r}(0,\theta)+r\frac{\partial^2u}{\partial r^2}(\eta,\theta)\right]\\ &\quad-\frac{1}{r^2}\left[\frac{\partial^2u}{\partial\theta^2}(0,\theta)+r\frac{\partial^3u}{\partial\theta^2\,\partial r}(0,\theta)\right.\\ &\quad\left.+\frac{r^2}{2}\frac{\partial^4u}{\partial\theta^2\,\partial r^2}(\xi,\theta)\right]\\ &\quad+\lambda u(r,\theta)=f(r,\theta)\end{aligned} \tag{3.3}$$

for any $r\in(0,1)$ and $\theta\in[0,2\pi)$, where $0<\eta$, $\xi<r$. Taking the limit as $r\to 0$ in (3.3) we obtain

$$O\left(\frac{1}{r^2}\right): \frac{\partial^2u}{\partial\theta^2}(0,\theta)=0, \tag{3.4}$$

$$O\left(\frac{1}{r}\right): \frac{\partial u}{\partial r}(0,\theta)+\frac{\partial^3u}{\partial\theta^2\,\partial r}(0,\theta)=0, \tag{3.5}$$

$$\begin{aligned}O(1):\ &-\left[2\frac{\partial^2u}{\partial r^2}(0,\theta)+\frac{1}{2}\frac{\partial^4u}{\partial\theta^2\,\partial r^2}(0,\theta)\right]\\ &+\lambda u(0,\theta)=f(0,\theta).\end{aligned} \tag{3.6}$$

By means of Fourier analysis we can show that (3.4) has one "component defect." In fact, substituting the Fourier expansion of $u(r,\theta)$,

$$u(r,\theta)=\sum_{n=-\infty}^{\infty}u_n^*(r)\,e^{in\theta}, \tag{3.7}$$

into (3.4) we have

$$-n^2u_n^*(0)=0, \qquad n=0,\pm 1,\ldots. \tag{3.8}$$

This condition gives no information concerning the component with $n=0$. The defect in the zeroth component of (3.4) should be redressed using the corresponding component from (3.5). This yields

$$(1-n^2)\frac{du_n^*}{dr}(0)=0 \qquad \text{with} \quad n=0,$$

or

$$\frac{du_0^*}{dr}(0)=0. \tag{3.9}$$

Making use of (3.7) we see that (3.9) may be written as

$$\int_0^{2\pi}\frac{\partial u}{\partial r}(0,\phi)\,e^{-in\phi}\,d\phi=0 \qquad \text{with} \quad n=0,$$

and if this is combined with (3.8) we obtain

$$\frac{\partial^2 u}{\partial \theta^2}(0, \theta) + \int_0^{2\pi} \frac{\partial u}{\partial r}(0, \phi)\, d\phi = 0. \tag{3.10}$$

The pole condition (3.10) will be used as a boundary condition at $r = 0$ in the pseudospectral approximation. It should be noted that (3.10) is equivalent to conditions (3.4.6) and (3.4.8) on page 90 of [3], but (3.10) is a more convenient formulation for use in the pseudospectral approximation.

4. PSEUDOSPECTRAL METHOD FOR POISSON-TYPE EQUATIONS ON THE UNIT DISK

The pseudospectral approximation to (3.1) using (3.10) as the boundary condition at $r = 0$ is described in Subsection 4.1. Numerical results are given in Subsection 4.2 and they are compared with those obtained by Eisen *et al.* [4]. Subsection 4.3 discusses the effect of finite-difference preconditioning. In [4], Eisen *et al.* considered a collocation approach, with basis functions especially chosen to permit the location of the coordinate singularity to be used as a collocation point. Basis functions were selected to maintain solution smoothness.

4.1. *Pseudospectral Method*

Let

$$r_k = \frac{1 - \cos(k\pi/N_r)}{2}, \qquad k = 0, ..., N_r, \tag{4.1}$$

$$\theta_j = \frac{\pi j}{N_\theta}, \qquad j = 0, ..., 2N_\theta - 1, \tag{4.2}$$

where N_r and N_θ are certain positive integers. Then the corresponding cardinal functions in r- and θ-directions, respectively, are [2]

$$h_k(r) = \prod_{\substack{l=0 \\ l \neq k}}^{N_r} \frac{r - r_l}{r_k - r_l}, \qquad k = 0, ..., N_r \tag{4.3}$$

and

$$C_j(\theta) = \frac{1}{2N_\theta} \sin[N_\theta(\theta - \theta_j)] \cot[0.5(\theta - \theta_j)]$$
$$= \frac{1}{2N_\theta} \sum_{n=-N_\theta}^{N_\theta} \frac{1}{\bar{c}_n} e^{in(\theta - \theta_j)},$$
$$j = 0, ..., 2N_\theta - 1, \tag{4.4}$$

where

$$\bar{c}_n = \begin{cases} 2, & n = \pm N_\theta \\ 1, & |n| \neq N_\theta. \end{cases} \tag{4.5}$$

If we approximate $u(r, \theta)$ by

$$u^c(r, \theta) = \sum_{k=0}^{N_r} \sum_{j=0}^{2N_\theta - 1} u_{kj} h_k(r)\, C_j(\theta), \tag{4.6}$$

then the collocation equations of (3.1) and (3.10) are

$$-\left[\frac{\partial^2 u^c}{\partial r^2}(r_l, \theta_m) + \frac{1}{r_l}\frac{\partial u^c}{\partial r}(r_l, \theta_m) + \frac{1}{r_l^2}\frac{\partial^2 u^c}{\partial \theta^2}(r_l, \theta_m)\right]$$
$$+ \lambda u^c(r_l, \theta_m) = f(r_l, \theta_m), \tag{4.7}$$

$$u^c(1, \theta_m) = g(\theta_m), \tag{4.8}$$

$$\frac{\partial^2 u^c}{\partial \theta^2}(0, \theta_m) + \int_0^{2\pi} \frac{\partial u^c}{\partial r}(0, \phi)\, d\phi = 0,$$
$$l = 1, ..., N_r - 1, \quad m = 0, ..., 2N_\theta - 1. \tag{4.9}$$

Equation (4.8) may be written simply as

$$u_{N_r m} = g(\theta_m), \qquad m = 0, ..., 2N_\theta - 1 \tag{4.10}$$

and (4.9) may be rewritten as

$$\frac{\partial^2 u^c}{\partial \theta^2}(0, \theta_m) + \frac{\pi}{N_\theta} \sum_{k=0}^{N_r} \sum_{j=0}^{2N_\theta - 1} u_{kj} \frac{dh_k}{dr}(0)$$
$$= 0, \qquad m = 0, ..., 2N_\theta - 1,$$

or

$$\frac{\partial^2 u^c}{\partial \theta^2}(0, \theta_m) - \frac{\partial^2 u^c}{\partial \theta^2}(0, \theta_{m+1})$$
$$= 0, \qquad m = 0, ..., 2N_\theta - 2,$$
$$\frac{\partial^2 u^c}{\partial \theta^2}(0, \theta_{2N_\theta - 1}) + \frac{\pi}{N_\theta} \sum_{k=0}^{N_r} \sum_{j=0}^{2N_\theta - 1} u_{kj} \tag{4.11}$$
$$\times \frac{dh_k}{dr}(0) = 0.$$

Hence u_{kj}, $k = 0, 1, ..., N_r$, $j = 0, ..., 2N_\theta - 1$ may be obtained by solving (4.7), (4.10), and (4.11). The matrix–vector multiplications in (4.7), (4.10), and (4.11) can be performed by fast transform or differentiation matrix methods and the reader may refer to [6].

TABLE I

Pseudospectral Methods for (3.1) with the Exact Solution $e^{r\cos\theta + r\sin\theta}$ and $\lambda = 0$

N	E_∞ Present	E_∞ [4][a]
2	2.775E − 2	9.589E − 2
3	7.789E − 3	9.310E − 2
4	5.105E − 4	2.185E − 2
5	5.819E − 5	2.365E − 3
6	3.956E − 6	6.037E − 5
7	3.486E − 7	2.777E − 5
8	2.611E − 8	3.272E − 6
9	1.609E − 9	1.902E − 7
10	8.573E − 11	2.040E − 9
11	4.838E − 12	8.545E − 10
12	2.421E − 13	6.817E − 11
16	1.601E − 14	4.767E − 12

[a] These values are quoted from Table II in [4].

4.2. *Numerical Experiments*

In order to compare the present results with those obtained by Eisen *et al.* [4], hereafter we shall take

$$N_r = N_\theta = N. \tag{4.12}$$

The algebraic system formed by (4.7), (4.10), and (4.11) for u_{kj}, $k = 0, ..., N$, $j = 0, ..., 2N-1$, is solved by the standard routine FO4ATF (Crout's factorisation method) from the NAG library on a VAX 8650 computer in double precision.

Tables I–III list present results and the best results obtained by Eisen *et al.* [4] for all examples considered there. In these tables we denote

$$E_\infty = \max_{\substack{k=0,...,N \\ j=0,...,2N-1}} |u^c(r_k, \theta_j) - u^e(r_k, \theta_j)|,$$

where $u^e(r, \theta)$ is the exact solution of (3.1). From these results we observe the exponential convergence of the method described in Subsection 4.1 for small N and infinitely differentiable (in polar coordinates) exact solution. It can be seen that the method described here is more accurate than that in [4]. Furthermore, the pole condition used here is represented entirely in physical space, unlike that in [4], so it is simpler to implement. Table III also shows that the present method may give good results for exact solutions which are not infinitely differentiable.

TABLE II

Pseudospectral Methods for (3.1) with the Exact Solution $\cos(3r\cos\theta + 4r\sin\theta + 0.7)$

λ	E_∞ Present $N=8$	Present $N=10$	[4][a] $N=8$
$-(2.4048)^2$	4.786E − 2	3.089E − 4	2.551E + 2
0	1.115E − 3	4.909E − 5	2.457E − 3
1	1.106E − 3	4.887E − 5	2.143E − 3
5	1.073E − 3	4.802E − 5	4.497E − 3
10	1.034E − 3	4.700E − 5	8.368E − 3
30	9.029E − 4	4.335E − 5	2.139E − 2
100	6.284E − 4	3.427E − 5	5.294E − 2

[a] These values, quoted from Table V in [4], were obtained by the even-parity method.

4.3. *Finite-Difference Preconditioning*

Using (3.10) we can also construct an effective finite-difference preconditioner for the pseudospectral differentiation matrix. The resulting preconditioned matrix has a low condition number which is independent of N, and the associated linear system could be solved conveniently by iterative methods. Furthermore, the preconditioning matrix is sparse, and the linear system could therefore be solved inexpensively. To construct the preconditioner we consider a finite difference approximation to (3.1) and (3.10). The discretisation of (3.1) is

$$\begin{aligned} &\frac{-2}{r_{k+1} - r_{k-1}} \left[\frac{(u_{k+1,j} - u_{k,j})}{(r_{k+1} - r_k)} - \frac{(u_{k,j} - u_{k-1,j})}{(r_k - r_{k-1})} \right] \\ &\quad - \frac{1}{r_k} \cdot \frac{(u_{k+1,j} - u_{k-1,j})}{(r_{k+1} - r_{k-1})} \\ &\quad - \frac{2}{r_k^2(\theta_{j+1} - \theta_{j-1})} \\ &\quad \times \left[\frac{(u_{k,j+1} - u_{k,j})}{(\theta_{j+1} - \theta_j)} - \frac{(u_{k,j} - u_{k,j-1})}{(\theta_j - \theta_{j-1})} \right] \\ &\quad + \lambda u_{k,j} = f_{k,j} \end{aligned} \tag{4.13a}$$

TABLE III

Pseudospectral Methods for (3.1) with $\lambda = 0$

$u^e(r, \theta)$	E_∞ $(N=8)$ Present	[4][a]
$e^{r\cos\theta + r\sin\theta}$	2.611E − 8	2.856E − 8
$\cos(7r\cos\theta + 8r\sin\theta + 0.7)$	0.411	1.474
	($N = 16$, 1.365E − 4)	($N = 16$, 4.873E − 4)
r^3	3.553E − 14	2.922E − 2
r^4	3.303E − 14	—
r^5	3.000E − 14	1.225E − 3
$r^{2.5}$	2.274E − 5	7.677E − 2
$r^{3.5}$	5.261E − 6	—
$r^{5.5}$	5.275E − 7	—

[a] These values, quoted from Table VI in [4], were obtained by the even-parity method.

TABLE IV

Eigenvalues of Minimum and Maximum Modulus for the Preconditioned Pseudospectral Differentiation Matrix $A_{FD}^{-1}A_{PS}$ with $N_r = N_\theta$

N_r	$2N_\theta$	$\|\lambda\|_{min}$	$\|\lambda\|^a_{max}$
2	4	1.000	$\pi^2/4$
3	6	0.955	$\pi^2/4$
4	8	0.968	$\pi^2/4$
5	10	0.964	$\pi^2/4$
6	12	0.966	$\pi^2/4$
7	14	0.964	$\pi^2/4$
8	16	0.964	$\pi^2/4$
9	18	0.963	$\pi^2/4$
10	20	0.963	$\pi^2/4$
11	22	0.962	$\pi^2/4$
12	24	0.962	$\pi^2/4$

[a] The computed values of $|\lambda|_{max}$ for all N_r and $2N_\theta$ listed agree with $\pi^2/4$ to at least 14 decimal places.

$$u_{N_r,j} = g(\theta_j), \qquad k = 1, ..., N_r - 1, \qquad j = 0, ..., 2N_\theta - 1, \tag{4.13b}$$

with the periodic condition

$$u_{k,2N_\theta} = u_{k,0}, \qquad k = 0, ..., N_r, \qquad \theta_{-1} := \theta_0 - (\theta_{2N_\theta} - \theta_{2N_\theta - 1}). \tag{4.14}$$

Equation (3.10) can be discretised as

$$\frac{2}{(\theta_{j+1}-\theta_{j-1})}\left[\frac{(u_{0,j+1}-u_{0,j})}{(\theta_{j+1}-\theta_j)} - \frac{(u_{0,j}-u_{0,j-1})}{(\theta_j-\theta_{j-1})}\right] + \sum_{i=0}^{2N_\theta-1}\frac{(\theta_{i+1}-\theta_i)}{2} \times \left[\frac{(u_{1,i}-u_{0,i})}{(r_1-r_0)} + \frac{(u_{1,i+1}-u_{0,i+1})}{(r_1-r_0)}\right] = 0,$$
$$j = 0, ..., 2N_\theta - 1,$$

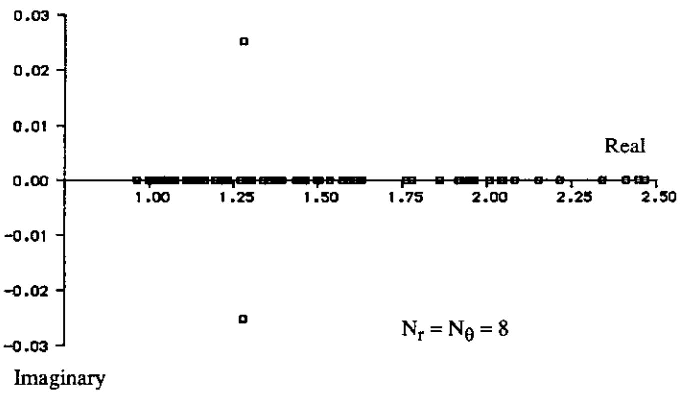

FIG. 4.1. Eigenvalues of the preconditioned pseudospectral differentiation matrix $(A_{FD})^{-1} A_{PS}$ with $N_r = N_\theta = 8$.

or

$$\frac{2}{(\theta_{j+1}-\theta_{j-1})}\left[\frac{(u_{0,j+1}-u_{0,j})}{(\theta_{j+1}-\theta_j)} - \frac{(u_{0,j}-u_{0,j-1})}{(\theta_j-\theta_{j-1})}\right] - \frac{2}{(\theta_{j+2}-\theta_j)} \times \left[\frac{(u_{0,j+2}-u_{0,j+1})}{(\theta_{j+2}-\theta_{j+1})} - \frac{(u_{0,j+1}-u_{0,j})}{(\theta_{j+1}-\theta_j)}\right] = 0,$$
$$j = 0, ..., 2N_\theta - 2,$$
$$\frac{2}{(\theta_{2N_\theta}-\theta_{2N_\theta-2})}\left[\frac{(u_{0,2N_\theta}-u_{0,2N_\theta-1})}{(\theta_{2N_\theta}-\theta_{2N_\theta-1})} - \frac{(u_{0,2N_\theta-1}-u_{0,2N_\theta-2})}{(\theta_{2N_\theta-1}-\theta_{2N_\theta-2})}\right] + \sum_{i=0}^{2N_\theta-1}\frac{(\theta_{i+1}-\theta_i)}{2} \times \left[\frac{(u_{1,i}-u_{0,i})}{(r_1-r_0)} + \frac{(u_{1,i+1}-u_{0,i+1})}{(r_1-r_0)}\right] = 0. \tag{4.15}$$

Incorporating (4.14), we see that (4.13) and (4.15) form a sparse algebraic system which can be solved efficiently by any sparse solver.

Denote the difference matrix arising from (4.13) and (4.15) by A_{FD} and the differentiation matrix arising from (4.7), (4.10), and (4.11) by A_{PS}. Table IV shows the eigenvalues of minimum and maximum modulus for the matrix $A_{FD}^{-1}A_{PS}$ when $N_r = N_\theta = N$. Figure 4.1 displays all of the eigenvalues of $A_{FD}^{-1}A_{PS}$ for $N_r = N_\theta = 8$. It is seen that the ratio $|\lambda|_{max}/|\lambda|_{min}$ is low and virtually independent of N. The matrix A_{FD} is therefore an effective preconditioner for the pseudospectral differentiation matrix A_{PS}.

5. CONCLUSIONS AND COMMENTS

By using the eigenvalue problem of Bessel's equation and Poisson-type equations on the unit disk, we have presented a simple method to construct a proper pole condition from the assumption of smoothness of the solution and from the differential equation itself. Numerical experiments show that by imposing the proper pole condition as the boundary condition at the coordinate singularity the standard pseudospectral method can capture fully the special behaviour of the solution near the singularity and it can obtain very high accuracy. Since the pole condition is simple and represented in the physical space and since the standard pseudospectral method is applied, fast transformation methods can be used. Consequently, the complexity of solving singular problems with smooth solutions by pseudospectral methods is no greater than that involved in solving non-singular problems with smooth solutions.

Although we have considered only two kinds of singular problems in this paper, we believe that the method presented is straightforward and that it could also be applied to other singular problems such as Poisson-type equations in cylindrical or spherical geometries. We propose to examine extensions of this type.

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